# Exercise Set #12
## "Discrete Mathematics" (2025)

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*Exercise 7 is to be submitted on Moodle before 23:59 on May 19th, 2025*

**E1.** Recall the definition of the Ramsey numbers: We define $R(k,l)$ to be the minimum number of vertices $n$ required to guarantee that any graph with $n$ vertices has a clique of size $k$ or an independent set of size $l$. Prove that $k \leq k'$ and $l \leq l'$ implies $R(k,l) \leq R(k',l')$.

**E2.** **(a)** Show that the Ramsey number $R(3,3)$ is 6.

**(b)** Let $G$ be a graph with at least $\binom{k+l-2}{k-1}$ vertices. Show that there is a clique of size $k$ or an independent set of size $l$.

**E3.** Let $|X| = n$ with $n \geq 2k$. Show that the Erdos-Ko-Rado theorem is sharp, i.e. that there exists an intersecting family $\mathcal{F}$ of $k$-element subsets of $X$ such that

$$|\mathcal{F}| = \binom{n-1}{k-1}$$

**E4.** Prove that each tournament has a Hamiltonian path.

**E5.** Prove that in any tournament there exists a vertex $v$ that can be reached from any other vertex by a directed path of length at most 2 .

**E6.** Let $K_n$ denote the complete graph on $n$ vertices. Suppose one colours all the edges of $K_n$ with one of two colours: red or blue.

**(a)** Let $v \in V(K_n)$ be a vertex. A bad cherry with vertex $v$ is a set of three vertices $u, v, w \in V(K_n)$ such that the colour of the edge $uv$ is different from the colour of the edge $vw$.

If $r(v)$ denotes the number of edges coming out of vertex $v$ which are painted red, show that the number of bad cherries with vertex $v$ is exactly $r(v)(n-1-r(v))$.

**(b)** We say that three vertices $u, v, w \in V(K_n)$ form a monochromatic triangle if the edges $uv, uw, vw$ all have the same colour. Show that, for any colouring of the edges of $K_n$ as above, the number of monochromatic triangles is at least

$$\frac{1}{4}\binom{n}{3} - n^2$$

**E7.** (**Exercise to submit**)

Show that every red-blue coloring of the edges of $K_{14}$ contains two disjoint red 4-cycles or two disjoint blue 4-cycles.

**Hint:** Prove that any 2-coloring of $K_6$ contains a monochromatic 4-cycle.